Efficient multigrid solvers for high-order discontinuous Galerkin discretizations via matrix-free operator evaluation

Martin Kronbichler, Peter Munch and Niklas Fehn

Institute for Computational Mechanics
Technical University of Munich
Garching b. München, Germany
e-mail: {kronbichler,munch,fehn}@lnm.mw.tum.de

Key Words: multigrid; discontinuous Galerkin method; high-order method; matrix-free method

ABSTRACT

My talk will present multigrid solvers for high-order discontinuous Galerkin (DG) discretizations of elliptic partial differential equations. The application background is the solution of the pressure Poisson equation in splitting methods for the incompressible Navier–Stokes equations, for which DG is the natural function space. To devise an efficient solver, we addressed three main challenges.

Firstly, global sparse matrices are no longer an efficient format for the matrix-vector product for polynomial degrees $p > 1$. This is due to the dense coupling between all shape functions on an element as well as its immediate neighbors [1]. A much faster realization is obtained by a matrix-free scheme that implements only the action of the matrix on a vector. In our case, the integrals underlying the DG discretization are computed on the fly by sum factorization techniques [2]. The design has been guided by high performance computing principles, taking into account the necessities of evolving exascale computer architecture. Indeed, the work [3] has shown that matrix-free ingredients in a multigrid solver allow for solving the Poisson equation for a moderately high degree $p = 4$ more quickly than a single matrix-vector product with a sparse matrix.

The second challenge is the definition of the multigrid levels, for which a combination of polynomial coarsening (p-multigrid), mesh coarsening (h-multigrid), auxiliary continuous finite element discretizations, and algebraic multigrid has been proposed in [1]. p-multigrid is necessary because a single high-order element can already contain hundreds of unknowns. Coarsening further with h-multigrid ensures excellent scalability to large-scale supercomputers with hundreds of thousands of compute cores, including the case of adaptively refined meshes [4], while algebraic multigrid as a coarse solver ensures the applicability to large unstructured meshes of complex geometries.

The third challenge is the definition of smoothers compatible with matrix-free operator evaluation. In our work, we found that polynomial smoothing with the Chebyshev iteration around the matrix diagonal or a tensor product approximation of a block-diagonal operator are highly efficient.

REFERENCES

[1] Fehn, N., Munch, P., Wall, W.A., and Kronbichler, M. Hybrid multigrid methods for high-order discontinuous Galerkin discretizations. *J. Comput. Phys.*, Vol. **415**, pp. 109539, (2020).

[2] Kronbichler, M. and Kormann, K. Fast matrix-free evaluation of discontinuous Galerkin finite element operators. *ACM Trans. Math. Softw.*, Vol. **45**(3), pp. 29:1–40, (2019).

[3] Kronbichler, M. and Ljungkvist, K. Multigrid for matrix-free high-order finite element computations on graphics processors. *ACM Trans. Parallel Comput.*, Vol. **6**(1), (2019).

[4] Clevenger, C.T., Heister, T., Kanschat, G., and Kronbichler, M. A flexible, parallel, adaptive geometric multigrid method for FEM. *ACM Trans. Math. Softw.*, Vol. **47**(1), pp. 7:1–27, (2020).